<u>Last week</u> Proven using Chern class

Counting multiplicity, a <u>general</u> cubic surface contains 27 lines

Know there are singular surfaces with ∞ lines.

<u>Question</u> do smooth cubic surfaces contain 27 lines?

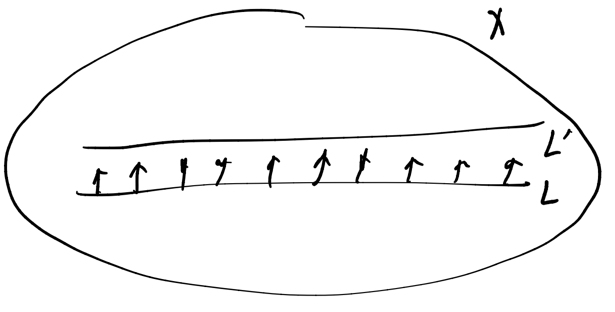
are these lines distinct?

<u>Approach</u> study the tangent space of $F_k(X)$

or more generally, the Hilbert scheme $H_P(X)$

Given line in surface

can move it using normal vectors.



<u>Theorem</u> Suppose $L \subseteq X$ is a k-plane in $X \subseteq \mathbb{P}^n$

Have $[L] \in F_k(X)$. the Zariski tangent space of $F_k(X)$ at $[L]$ is $H^0(N_{L/X})$

What is $N_{Y/X}$?

Defn Y nonsing subvar of X nonsing

then $N_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$ is the

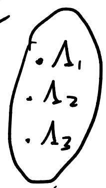
normal sheaf (on Y)

Fact: locally free of rank $\operatorname{codim}(Y, X)$.

Proof of theorem use deformation thy.

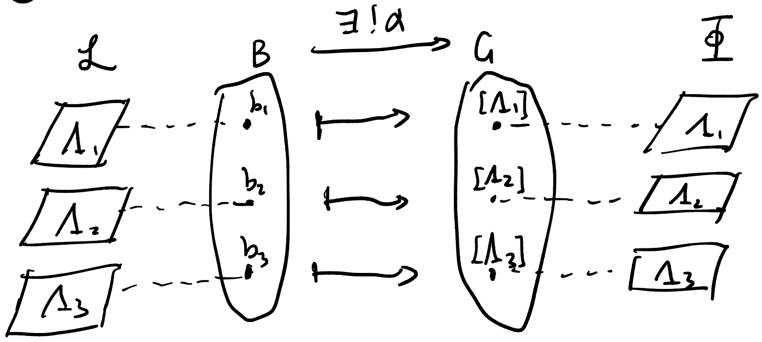
We work more generally on Hilbert schemes.

First note Grassmannian can be characterized by

Universal property $\quad G = \{k \text{ planes in } \mathbb{P}V\} =$ 

$\exists\ \Phi \subseteq G \times \mathbb{P}V \longrightarrow G$ tautological bundle whose fiber at Λ is $\{\Lambda\} \times \Lambda$

s.t. $\forall\ \mathcal{L} \subseteq B \times \mathbb{P}V$
$\downarrow$
B $\quad$ flat families of k-planes.



we have $\mathcal{L} = \alpha^* \Phi$

Generalise to Hilbert scheme

$X \subseteq \mathbb{P}^n$ closed subscheme $P(t)$ polynomial

$H_P(X)$ Hilbert scheme moduli of subschemes of X with Hilbert polynomial P.

Given by universal property of tautological family

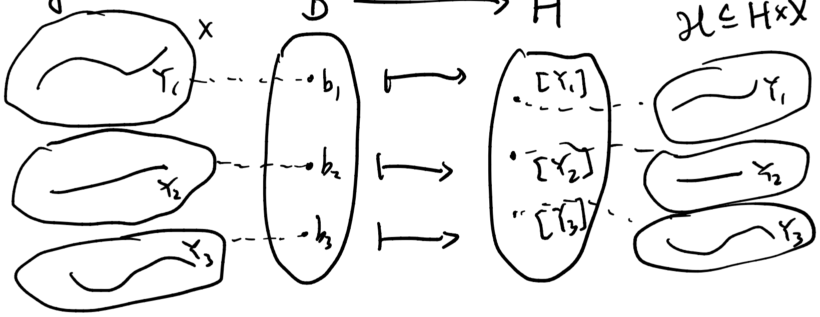
$\exists\ \mathcal{H} \subseteq Hilb(X) \times X$

$$\begin{array}{c} \mathcal{H} \\ \downarrow \\ Hilb(X) \end{array}$$ with fiber over $Y = \{Y\} \times Y$.

$\forall\ \mathcal{Y} \subseteq B \times X$

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ B \end{array}$$ flat family of subschemes $Y \subseteq X$ with Hilbert polynomial P,

$B \xrightarrow{\exists ! \alpha} H$



s.t. $\mathcal{Y} = \alpha^* \mathcal{H}$.

Hilb poly of Y is dim of degree t part of homogeneous coordinate ring.

$$t \mapsto \dim_{\mathbb{C}} \left(\mathbb{C}[x_0, \ldots, x_n] / I_Y \right)_{\deg = t}$$

When $P(t) = \binom{t+k}{k}$, $H_P(X) = F_k(X)$

Now for deformation theory

A <u>deformation of $Y \subseteq X$ along T</u> at $0 \in T$

$\mathcal{Y} \subseteq T \times X \to T$ flat with $\mathcal{Y}_0 = Y$

A deformation is <u>first order</u>

if $T = T_m = \operatorname{Spec} k[\varepsilon_1, \dots, \varepsilon_m]/(\varepsilon_1, \dots, \varepsilon_m)^2$ for some m.

= fat point of dim m.

$\overset{\text{universal property}}{\Longrightarrow}$ {deformations of $Y \subseteq X$ over T at 0}

$\longleftrightarrow \quad \operatorname{Mor}_{[Y]}(T, H)$

We want to consider $T = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$

because $\operatorname{Mor}_{[Y]}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, H) \cong T_{[Y]} H$ is Zariski tangent space.

<u>Thm</u> $\exists$ bijective correspondence

{deformation along T_1} $\longleftrightarrow \operatorname{Hom}_{\mathcal{O}_X}(I_Y/I_Y^2, \mathcal{O}_Y)$

(get $T_{[Y]} H \leftrightarrow H^0(N_{Y/X})$)

<u>Lemma</u> (characterization of flat morphisms)

R-mod M is flat iff the multiplication map

$I \otimes M \to IM$ is isomorphism $\forall$ ideal I.

Cor   Spec $k[\varepsilon]/(\varepsilon^2)$ - module M is flat iff multiplication map

$$M \xrightarrow{\cdot \varepsilon} M \quad \text{induces isomorphism}$$

$$M/\varepsilon M \cong \varepsilon M$$

Pf of Thm   Assume X, Y affine.

Let $\varphi : I_Y / I_Y^2 \longrightarrow \mathcal{O}_Y$

Define $I_\varphi := \{ g + h\varepsilon \mid g \in I_Y,\ h \in \mathcal{O}_X \text{ s.t. } h = \varphi(g) \bmod I_Y \}$

$\subseteq \mathcal{O}_X \otimes \operatorname{Spec} k[\varepsilon]/\varepsilon^2$

Let $\mathcal{Y} \subseteq X \times T$ be cut out by I_φ

then $\mathcal{Y}$ is a family over T with central fiber

$\mathcal{Y}_0 = Y$ (set $\varepsilon = 0$)

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \mathcal{Y} \arrow[d] \\ 0 \arrow[r, hook] & T \end{tikzcd}$$

Note $0 + h\varepsilon \in I_\varphi \iff h \in I_Y$

so $I_Y \cap \varepsilon \mathcal{O}_X = \varepsilon \cdot I_Y$

Hence

$$\begin{aligned} (\varepsilon) \cdot \mathcal{O}_{\mathcal{Y}} &= \varepsilon \mathcal{O}_X / I_\varphi \\ &= \varepsilon \mathcal{O}_X / I_\varphi \cap \varepsilon \mathcal{O}_X \\ &= \varepsilon \mathcal{O}_X / \varepsilon I_Y \\ &= \mathcal{O}_Y = (\mathcal{O}_{\mathcal{Y}} / \varepsilon \cdot \mathcal{O}_{\mathcal{Y}}) \end{aligned}$$

$\Rightarrow \mathcal{Y}$ is flat over T so is deformation

Conversely, given $Y \subseteq T \times X$ flat over T

say defined by $I \subseteq \mathcal{O}_X[\varepsilon]$

If central fiber is Y, then $I = I_Y \bmod \varepsilon$.

So $I \cap \mathcal{O}_X \supseteq I_Y$ so $\forall g \in I_Y, \exists g + h\varepsilon \in I$.

by flatness, $I \cap \varepsilon \cdot \mathcal{O}_X = \varepsilon \cdot I_Y$

Therefore if $g + h\varepsilon$ and $g + h'\varepsilon$ in I

then $h - h' \in I_Y$

Get well-defined morphism

$$\varphi : I_Y / I_Y^2 \to \mathcal{O}_X / I_Y = \mathcal{O}_Y$$

$$g \mapsto h \quad \text{for } g + h\varepsilon \in I$$

Thm The above identification
$T_{[Y]}H \cong H^0(N_{Y/X})$ is isom of vector spaces

Pf Need vector space structure on deformations.

We do scalar multiplication. addition is covered in text.

Let $\varphi : \operatorname{Spec}[\varepsilon]/\varepsilon^2 \to H$

define $a\varphi : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varphi} H$

induced by $k[\varepsilon]/\varepsilon^2 \to k[\varepsilon]/\varepsilon^2$

$$\varepsilon \mapsto a\varepsilon$$

This is the point in $T_{[Y]}H = (m/m^2)^*$ given by

$$m/m^2 \xrightarrow{\varphi} \varepsilon\cdot k \cong k \xrightarrow{\cdot a} \varepsilon\cdot k \cong k$$

$$= a\cdot\varphi$$

so compatible with $T_{[Y]}H$

Let $\mathcal{Y} = \varphi^*\mathcal{H}$

$$= \varphi^{-1}\mathcal{H} \otimes_{\varphi^{-1}\mathcal{O}_H} \mathcal{O}_T$$

If take $a\cdot\varphi$ instead, then the map

$(a\varphi)^{-1}\mathcal{O}_H \to \mathcal{O}_T$ is given by

$$\varphi^{-1}\mathcal{O}_H \to \mathcal{O}_T \xrightarrow{\varepsilon \mapsto a\varepsilon} \mathcal{O}_T$$

Hence resulting map $I_Y/I_Y^2 \to \mathcal{O}_Y$

$g \mapsto h$

becomes $g \mapsto ah$

Ex when Y Cartier divisor. (= Weil for smooth (⇒ regular))

$$\mathcal{N}_{Y/X} = \mathcal{O}_X(Y).$$

For lines in cubic surface. $\mathcal{N}_{L/X} = \mathcal{O}_X(L)|_L$

$= \mathcal{O}_L(-1)$ since self intersection is in general $2-d$.

$\Rightarrow H^0(N_{L/X}) = 0$, $F_1(X)$ is 27 discrete points.